Lecture Notes for Chapter 1: Mathematical Induction

The principle of mathematical induction is used to prove that a given proposition (formula, equality, inequality, and more) is true for all positive integer numbers greater than or equal to some integer N.

Let us denote the proposition in question by P(n), where n is a positive integer. The proof involves two steps:
Step 1: We first establish that the proposition P(n) is true for the lowest possible value of the positive integer n.
Step 2: We assume that P(k) is true and establish that P(k+1) is also true.
Example 1- Use mathematical induction to prove that

$$1 + 2 + 3 + ... + n = n(n + 1)/2$$

for all positive integers n.
Solution to Example 1:
Let the statement P(n) be

$$1 + 2 + 3 + ... + n = n(n + 1)/2$$

STEP 1: We first show that p(1) is true.
Left Side = 1
Right Side = 1 (1 + 1) / 2 = 1
Both sides of the statement are equal hence p(1) is true.
STEP 2: We now assume that p(k) is true

$$1 + 2 + 3 + ... + k = k(k + 1)/2$$

and show that p(k + 1) is true by adding k + 1 to both sides of the above statement

$$1 + 2 + 3 + ... + k + (k + 1) = k(k + 1)/2 + (k + 1)$$

$$= (k + 1)(k/2 + 1)$$

$$= (k + 1)(k + 2)/2$$

The last statement may be written as

$$1 + 2 + 3 + ... + k + (k + 1) = (k + 1)(k + 2)/2$$

Which is the statement p(k + 1).
Example 2- Use mathematical induction to prove that

$$1^2 + 2^2 + 3^2 + ... + n^2 = n(n + 1)(2n + 1)/6$$

For all positive integers n.

Solution to Example 2:
Statement P(n) is defined by

$$1^2 + 2^2 + 3^2 + ... + n^2 = n(n + 1)(2n + 1)/6$$

STEP 1: We first show that p(1) is true. Left Side = 1^2 = 1 Right Side = 1 (1 + 1) (2*1 + 1)/ 6 = 1 Both sides of the statement are equal hence p(1) is true.
STEP 2: We now assume that p (k) is true

$$1^2 + 2^2 + 3^2 + ... + k^2 = k(k + 1)(2k + 1)/6$$

and show that p(k + 1) is true by adding $(k + 1)^2$ to both sides of the above statement

$$1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 = k(k + 1)(2k + 1)/6 + (k + 1)^2$$

Set common denominator and factor k + 1 on the right side

$$= (k + 1)[k(2k + 1) + 6(k + 1)]/6$$

Expand k (2k + 1)+ 6 (k + 1)

$$= (k + 1)[2k2 + 7k + 6]/6$$

Now factor $2k^2$ + 7k + 6.

$$= (k + 1)[(k + 2)(2k + 3)]/6$$

We have started from the statement P(k) and have shown that

$$1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 = (k + 1)[(k + 2)(2k + 3)]/6$$

Which is the statement P(k + 1).
Example 3- Use mathematical induction to prove that for any positive integer number n

$$n^3 + 2n \text{ is divisible by } 3$$

Solution to Example 3:
Statement P(n) is defined by

$$n^3 + 2n \text{ is divisible by } 3$$

STEP 1: We first show that p(1) is true. Let n = 1 and calculate $n^3 + 2n$

$$1^3 + 2(1) = 3$$

3 is divisible by 3. Hence p(1) is true.
STEP 2: We now assume that p(k) is true $k^3 + 2k$ is divisible by 3
is equivalent to
$k^3 + 2k = 3M$, where M is a positive integer.

We now consider the algebraic expression $(k+1)^3+2(k+1)$; expand it and group like terms

$$(k+1)^3+2(k+1) = k^3+3k^2+5k+3$$

$$= [k^3+2k]+[3k^2+3k+3]$$

$$= 3M+3[k^2+k+1] = 3[M+k^2+k+1]$$

Hence $(k+1)^3+2(k+1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

Example 4- Show that $n! > 3^n$ for $n \geq 7$.

Solution to Example 4:

For any $n \geq 7$, let P(n) be the statement that $n! > 3^n$.

Base Case. The statement P(7) says that $7! = 7\times6\times5\times4\times3\times2\times1 = 5040 > 3^7 = 2187$, which is true.

Inductive Step. Fix $k \geq 7$, and suppose that P(k) holds, that is, $k! > 3^k$.

It remains to show that P(k+1) holds, that is, that $(k+1)! > 3^{(k+1)}$.

$$(k+1)! = (k+1)k! > (k+1)3^k \geq (7+1)3^k = 8\times3^k > 3\times3^k = 3^{(k+1)}.$$

Therefore P(k+1) holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P(n) holds.

Example 5- Show that $3^n > n^2$.

Solution to Example 5:

STEP 1: We first show that p (1) is true. Let n = 1 and calculate 3^1 and 1^2 and compare them

$3^1 = 3$

$1^2 = 1$

3 is greater than 1 and hence p(1) is true.

Let us also show that P(2) is true.

$3^2 = 9$

$2^2 = 4$

Hence P(2) is also true.

STEP 2: We now assume that p(k) is true

$$3^k > k^2$$

Multiply both sides of the above inequality by 3

$$3\times3^k > 3\times k^2$$

The left side is equal to 3^{k+1}. For $k > 2$, we can write

$$k^2 > 2k \quad and \quad k^2 > 1$$

We now combine the above inequalities by adding the left hand sides and the right hand sides of the two inequalities

$$2k^2 > 2k + 1$$

We now add k^2 to both sides of the above inequality to obtain the inequality

$$3k^2 > k^2 + 2k + 1$$

Factor the right side we can write

$$3 \times k^2 > (k+1)^2$$

If $3 \times 3^k > 3 \times k^2$ and $3 \times k^2 > (k+1)^2$ then

$$3 \times 3^k > (k+1)^2$$

Rewrite the left side as 3^{k+1}

$$3^{k+1} > (k+1)^2$$

Which proves that P(k + 1) is true.

1 Exercises for Chapter 1: Mathematical Induction

1.1 Summations

Prove by induction on $n \geq 0$ that

1. $1 + 3 + 5 + ... + (2n - 1) = n^2$.

2. $\sum_{i=1}^{n} i = n(n+1)/2$.

3. $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

4. $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$.

5. $\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$.

6. $\sum_{i=1}^{n} i(i+1)(i+2) = n(n+1)(n+2)(n+3)/4$.

7. $\sum_{i=1}^{n} i.i! = (n+1)! - 1$.

8. $\sum_{i=1}^{n} 1/2^i = 1 - 1/2^n$.

9. $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$.

10. Prove by induction on $n \geq 1$ that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

11. Prove by induction on $n \geq 1$ that for every $a \neq 1$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

1.2 INEQUALITIES

12. Prove by induction on $n \geq 3$ that if $2n + 1 \leq 2^n$.

13. Prove by induction on $n \geq 1$ that if $x > -1$, $(1+x)^n \geq 1 + nx$.

14. Prove by induction on $n \geq 7$ that $3^n < n!$.

15. Prove by induction on $n \geq 5$ that $2^n > n^2$.

16. Prove by induction on $k \geq 1$ that $\sum_{i=1}^{n} i^k \leq n^k(n+1)/2$.

17. Prove by induction on $n \geq 0$ that

$$\sum_{i=1}^{n} \frac{1}{i^2} < 1 - \frac{1}{n}$$

1.3 FLOORS AND CEILINGS

Suppose $x \in \mathbb{R}^+$. The floor of x, denoted $\lfloor x \rfloor$ is defined to be the largest integer that is no larger than x. The ceiling of x, denoted $\lceil x \rceil$ is defined to be the smallest integer that is no smaller than x.

18. Prove by induction on $n \geq 0$ that

$$\lfloor \frac{n}{2} \rfloor = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

19. Prove by induction on $n \geq 0$ that

$$\lceil \frac{n}{2} \rceil = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

20. Prove by induction on $n \geq 1$ that for all $m \in \mathbb{R}^+$

$$\lceil \frac{n}{m} \rceil = \lfloor \frac{n+m-1}{m} \rfloor.$$

1.4 DIVISIBILITY

21. Prove by induction on $n \geq 1$ that $7^n - 1$ is divisible by 6.

22. Prove by induction on $n \geq 0$ that $n^3 + 2n$ is divisible by 3.

23. Prove by induction on $n \geq 0$ that $n^5 - n$ is divisible by 5.

24. Prove by induction on $n \geq 0$ that $5^{n+1} + 2.3^n + 1$ is divisible by 8.

25. Prove by induction that a decimal number is divisible by 3 iff the sum of its digits is divisible by 3.

26. Prove by induction that a decimal number is divisible by 9 iff the sum of its digits is divisible by 9.

27. Prove by induction that the sum of the cubes of three successive natural numbers is divisible by 9.

1.5 POSTAGE STAMPS

28. Show that any integer postage greater than 7 cents can be formed by using only 3-cent and 5-cent stamps.

29. Show that any integer postage greater than 34 cents can be formed by using only 5-cent and 9-cent stamps.

30. Show that any integer postage greater than 5 cents can be formed by using only 2-cent and 7-cent stamps.

31. Show that any integer postage greater than 59 cents can be formed by using only 7-cent and 11-cent stamps.

32. What is the smallest value of k such that any integer postage greater than k cents can be formed by using only 4-cent and 9-cent stamps? Prove your answer correct.

33. What is the smallest value of k such that any integer postage greater than k cents can be formed by using only 6-cent and 11-cent stamps? Prove your answer correct.

1.6 FIBONACCI NUMBERS

The Fibonacci numbers F_n for $(n \geq 0)$ are defined recursively as follows: $F_0 = 0$, $F_1 = 1$,and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

34. Prove by induction on n that $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

35. Prove by induction that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$.

36. Prove by induction on $n \geq 1$ that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$.

37. Prove by induction on $n \geq 1$ that $F_{n+1} F_{n+2} = F_n F_{n+3} + (-1)^n$.

38. Prove by induction on $n \geq 2$ that $F_{n-1} F_{n+1} = F_n^2 + (-1)^n$.

1.7 BINOMIAL COEFFICIENTS

The binomial coefficient

$$\binom{n}{r},$$

For $n, r \in N, r \leq n$ is defined to be the number of ways of choosing r things from n without replacement, that is,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

39. Prove by induction on n that

$$\sum_{m=0}^{n} \binom{n}{m} = 2^n$$

40. Prove by induction on $n \geq 1$ that for all $1 \leq m \leq n$,

$$\binom{n}{m} \leq n^m$$

41. Prove by induction that for all $n \in N$

$$\sum_{i=0}^{n} i.\binom{n}{i} = n.2^{n-1}$$

1.8 Miscellaneous

42. Using induction, show that n straight lines separate the plane into $(n^2 + n + 2)/2$ regions. Suppose no two lines are parallel and no three lines have a common point.

43. Prove that n straight lines in the plane, all passing through a single point, divide the plane into 2n regions.

44. Prove by induction that a tree with n vertices has exactly n - 1 edges.

45. Prove by induction that a complete binary tree with n levels has $2^n - 1$ vertices.

46. Prove by induction that if there exists an n-node tree in which all the nonleaf nodes have k children, then n mod k = 1.

47. Prove by induction that for every n such that n mod k = 1, there exists an n-node tree in which all of the nonleaf nodes have k children.

48. Prove by induction that between any pair of vertices in a tree there is a unique path.

49. Prove that any set of regions defined by n lines in the plane can be colored with two colors so that no two regions that share an edge have the same color.

50. One of the interesting problems of algorithm design is the problem of tiling or to tile the given floor/board (Figure 1) using 1×m tiles.

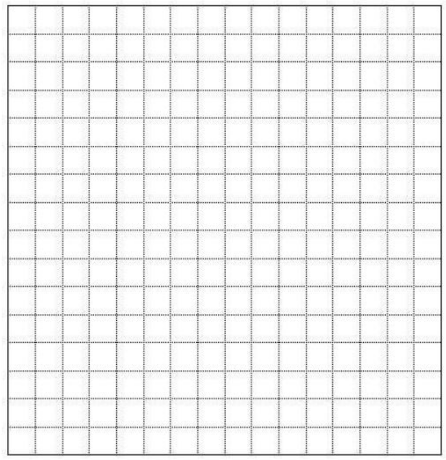

Figure 1: A sample of floor/board

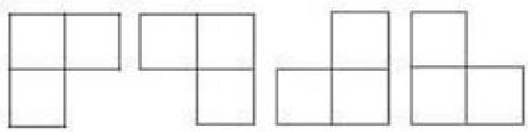

Figure 2: L shaped tiles

The purpose is tiling this plot of land using tiles with the following shapes (L shaped tile, where a L shaped tile is a 2×2 square with one cell of size 1×1 missing.), as shown in Figure 2.

In such a way that one of the checkered earth houses is not covered. It can be assumed that this house will be used to build a garden or a small pond. Note that the size of the sides of the small squares of the tiles is the same as the checkered plate above one meter. We also do not have the right to break these tiles into smaller pieces. For example, suppose in Figure 3 we want the specified house not to be covered:

This floor can be tiled as Figure 4. Given a n by n board where n is of form 2^k where $k \geq 1$ (basically n is a power of 2 with minimum value as 2). The board has one missing cell (of size 1×1). Prove by induction that this board can be tilled by the tiles shown in Figure 2.

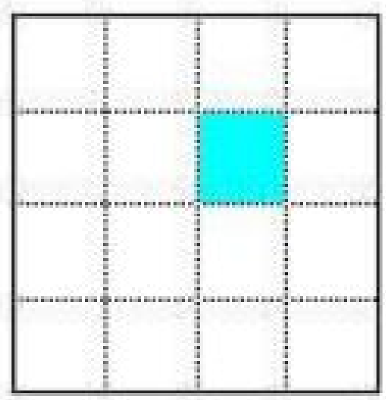

Figure 3: A sample of floor/board

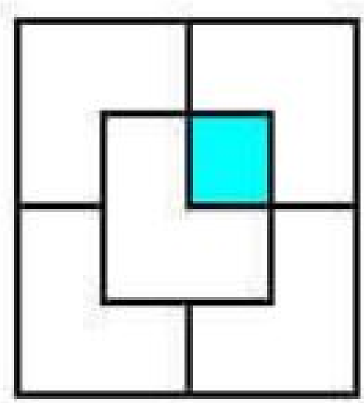

Figure 4: A tiled board